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$\mathcal{L}_1$ Adaptive Output-Feedback Controller for Non-Strictly-Positive-Real Reference Systems: Missile Longitudinal Autopilot Design

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This paper presents an extension of the $\mathcal{L}_1$ adaptive output-feedback controller to systems of unknown relative degree in the presence of time-varying uncertainties without restricting the rate of their variation. As compared with earlier results in this direction, a new piecewise continuous adaptive law is introduced, along with the low-pass-filtered control signal that allows for achieving arbitrarily close tracking of the input and the output signals of the reference system, the transfer function of which is not required to be strictly positive real. Stability of this reference system is proved using a small-gain-type argument. The performance bounds between the closed-loop reference system and the closed-loop $\mathcal{L}_1$ adaptive system can be rendered arbitrarily small by reducing the step size of integration. Missile longitudinal autopilot design is used as an example to illustrate the theoretical results.

I. Introduction

MODERN fighter aircraft and munitions need to operate in highly nonlinear and uncertain flight regimes, for which accurate aerodynamic modeling is not possible or is overly expensive. The control design challenge for these vehicles is to ensure safe operation in the presence of such uncertainties, without sacrificing the maneuverability of the vehicle. Both classical and modern control design methods have been extensively investigated for performance improvement of such vehicles in the presence of uncertainties. Their performance is well known to depend upon accurate knowledge of the system dynamics and is limited by the presence of unmodeled high-frequency effects. The same is particularly true for adaptive methods that lose their robustness in the presence of fast adaptation [1–4]. The recently developed $\mathcal{L}_1$ adaptive controller [5–7] opened new opportunities for addressing the control challenge for highly uncertain vehicles in the presence of component failures, aerodynamic uncertainties, and disturbances. It is important at this point to emphasize the significance of output-feedback architectures, which have the potential to relax the matching assumption, typically appearing in state-feedback adaptive control architectures. One of the fundamental limitations of adaptive output-feedback approaches is the requirement for the reference-system dynamics to have a strictly-positive-real (SPR) transfer function between the input and the regulated output [8]. When the SPR condition does not hold, then the obtained results lead to ultimate boundedness, which are hard to quantify rigorously and/or to reduce in desirable way [9,10]. On the other hand, the SPR requirement for the reference system restricts the class of reference behaviors that can be achieved by adaptation for the given unknown plant [6,11].

This paper extends the results of [6] to an output-feedback framework by considering reference systems that do not verify the SPR condition for their input–output transfer function. The key difference from the results in [6] is the new piecewise continuous adaptive law. The adaptive control is defined as the output of a low-pass filter, resulting in a continuous signal despite the discontinuity of the adaptive law. Similar to [6], the $\mathcal{L}_\infty$-norms of both input/output error signals between the closed-loop adaptive system and the reference system can be rendered arbitrarily small by reducing the step size of integration.

We note that adaptive algorithms achieving arbitrarily improved transient performance for a system's output were reported in [12–26]. As compared with those results, [5,6] presented the opportunity to also regulate the performance bound for a system's input signal by rendering it arbitrarily close to the corresponding signal of a bounded linear time-invariant reference system. Unlike conventional adaptive controllers, the $\mathcal{L}_1$ adaptive controllers adapt fast, leading to desired transient and asymptotic tracking with a guaranteed, bounded away from zero, time-delay margin [27]. The results from [5,6] have been intensively applied in flight tests [11,28–31] and various mid- to high-fidelity simulation environments [32–35]. Insights into the performance of $\mathcal{L}_1$ adaptive controller can be obtained from the analysis of a simple linear system in [36], in which sensitivity and cosensitivity transfer functions are analyzed for disturbance rejection and noise tolerance in the presence of a large adaptation rate. Further, the results in [37] provide systematic design guidelines for selection of the underlying filter to achieve the desired performance bound while retaining a guaranteed time-delay margin.

This paper presents the adaptive output-feedback counterpart of the results in [5,6], without enforcing the SPR condition on the input/output transfer function of the desired reference system, which typically appears in conventional adaptive output-feedback schemes. The paper also uses the longitudinal dynamics of the missile from [38] to demonstrate the usefulness of the methodology. Different applications of this methodology can be found in [39,40].

The paper is organized as follows. Section II gives the problem formulation. In Sec. III, the closed-loop reference system is introduced. In Sec. IV, some preliminary results are developed toward the definition of the $\mathcal{L}_1$ adaptive controller. In Sec. V, the novel $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform performance bounds are presented in Sec. VI. In Sec. VII, simulation

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results are presented, and Sec. VIII concludes the paper. The small-gain theorem and some basic definitions from linear systems theory used throughout the paper are given in the Appendix. Unless otherwise mentioned, $\|\cdot\|$ will be used for the 2-norm of the vector.

II. Problem Formulation

Consider the following single-input/single-output (SISO) system:

$$y(s) = A(s)(u(s) + d(s)), \qquad y(0) = 0 \tag{1}$$

where $u(t) \in \mathbb{R}$ is the input; $y(t) \in \mathbb{R}$ is the system output; $A(s)$ is a strictly proper unknown transfer function of unknown relative degree n_r, for which only a known lower bound $1 < d_r \le n_r$ is available; $d(s)$ is the Laplace transform of the time-varying uncertainties and disturbances $d(t) = f(t, y(t))$ and f is an unknown map, subject to the following assumptions.

Assumption 1. There exist constants $L > 0$ and $L_0 > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|, \qquad |f(t, y)| \le L|y| + L_0$$

hold uniformly in $t \ge 0$, where the numbers L and L_0 can be arbitrarily large.

Assumption 2. There exist constants $L_1 > 0$, $L_2 > 0$, and $L_3 > 0$ such that

$$|\dot{d}(t)| \le L_1|\dot{y}(t)| + L_2|y(t)| + L_3$$

for all $t \ge 0$, where the numbers L_1, L_2, and L_3 can be arbitrarily large.

Let $r(t)$ be a given bounded continuous reference input signal. The control objective is to design an adaptive output-feedback controller $u(t)$ such that the system output $y(t)$ tracks the reference input $r(t)$ following a desired reference model: that is,

$$y(s) \approx M(s)r(s)$$

where $M(s)$ is a minimum-phase stable transfer function of relative degree d_r. We rewrite the system in Eq. (1) as

$$y(s) = M(s)(u(s) + \sigma(s)), \qquad y(0) = 0 \tag{2}$$

$$\sigma(s) = ((A(s) - M(s))u(s) + A(s)d(s))/M(s) \tag{3}$$

Let (A_m, b_m, c_m) be the minimal realization of $M(s)$; that is, it is controllable and observable and A_m is Hurwitz. The system in Eq. (2) can be rewritten as

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + b_m(u(t) + \sigma(t))\\ y(t) &= c_m^\top x(t), \qquad x(0) = x_0 = 0\end{aligned} \tag{4}$$

III. Closed-Loop Reference System

Consider the following closed-loop reference system:

$$y_{\mathrm{ref}}(s) = M(s)(u_{\mathrm{ref}}(s) + \sigma_{\mathrm{ref}}(s)) \tag{5}$$

$$\sigma_{\mathrm{ref}}(s) = \frac{(A(s) - M(s))u_{\mathrm{ref}}(s) + A(s)d_{\mathrm{ref}}(s)}{M(s)} \tag{6}$$

$$u_{\mathrm{ref}}(s) = C(s)(r(s) - \sigma_{\mathrm{ref}}(s)) \tag{7}$$

where $d_{\mathrm{ref}}(t) = f(t, y_{\mathrm{ref}}(t))$, and $C(s)$ is a strictly proper system of relative order d_r, with its dc gain $C(0) = 1$. Further, the selection of $C(s)$ and $M(s)$ must ensure that

$$H(s) = A(s)M(s)/(C(s)A(s) + (1 - C(s))M(s)) \tag{8}$$

is stable and

$$\|G(s)\|_{\mathcal{L}_1} L < 1 \tag{9}$$

where $G(s) = H(s)(1 - C(s))$. This in turn restricts the class of systems $A(s)$ in Eq. (1), for which the proposed approach in this paper can achieve stabilization. However, as discussed later in Remark 3, the class of such systems is not empty. Letting

$$A(s) = \frac{A_n(s)}{A_d(s)}, \qquad C(s) = \frac{C_n(s)}{C_d(s)}, \qquad M(s) = \frac{M_n(s)}{M_d(s)} \tag{10}$$

where the numerators and the denominators are all polynomials of s. It follows from Eq. (8) that

$$H(s) = \frac{C_d(s)M_n(s)A_n(s)}{H_d(s)} \tag{11}$$

where

$$H_d(s) = C_n(s)A_n(s)M_d(s) + M_n(s)A_d(s)(C_d(s) - C_n(s)) \tag{12}$$

A strictly proper $C(s)$ implies that the order of $C_d(s) - C_n(s)$ and $C_d(s)$ is the same. Because the order of $A_d(s)$ is higher than that of $A_n(s)$, the transfer function $H(s)$ is strictly proper.

Lemma 1. If $C(s)$ and $M(s)$ verify the condition in Eq. (9), the closed-loop reference system in Eqs. (5–7) is stable.

Proof. It follows from Eqs. (6) and (7) that

$$u_{\mathrm{ref}}(s) = \frac{C(s)M(s)r(s) - C(s)A(s)d_{\mathrm{ref}}(s)}{C(s)A(s) + (1 - C(s))M(s)} \tag{13}$$

From Eqs. (5) and (6), one can derive

$$y_{\mathrm{ref}}(s) = H(s)(C(s)r(s) + (1 - C(s))d_{\mathrm{ref}}(s)) \tag{14}$$

Because $H(s)$ is strictly proper and stable,

$$G(s) = H(s)(1 - C(s))$$

is also strictly proper and stable; therefore,

$$\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \|H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}(L\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty} + L_0)$$

Using the condition in Eq. (9), one can write

$$\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \rho, \qquad \rho = \frac{\|H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}L_0}{1 - \|G(s)\|_{\mathcal{L}_1}L} < \infty \tag{15}$$

Hence, $\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty}$ is bounded, and the proof is complete. □

IV. Preliminaries for the Main Result

Let

$$H_0(s) = A(s)/(C(s)A(s) + (1 - C(s))M(s)) \tag{16}$$

$$H_1(s) = ((A(s) - M(s))C(s))/(C(s)A(s) + (1 - C(s))M(s)) \tag{17}$$

$$H_2(s) = H(s)C(s)/M(s) \tag{18}$$

$$H_3(s) = -(M(s)C(s))/((C(s)A(s) + (1 - C(s))M(s))) \tag{19}$$

Using the expressions from Eqs. (10) and (12),

$$H_0(s) = C_d(s)A_n(s)M_d(s)/H_d(s)$$

and $H_1(s)$ can be rewritten as

$$H_1(s) = (C_n(s)A_n(s)M_d(s) - C_n(s)A_d(s)M_n(s))/H_d(s) \tag{20}$$

Because the degree of $C_d(s) - C_n(s)$ is larger than $C_n(s)$ by d_r, the degree of

$$M_n(s)A_d(s)(C_d(s) - C_n(s))$$

is larger than

$$C_n(s)A_d(s)M_n(s)$$

by d_r. Because the degree of $A_d(s)$ is larger than $A_n(s)$ by $\geq d_r$ and the degree of $M_n(s)$ is larger than $M_d(s)$ by d_r, the degree of

$$M_n(s)A_d(s)(C_d(s) - C_n(s))$$

is larger than that of

$$C_n(s)A_n(s)M_d(s)$$

Therefore, $H_1(s)$ is strictly proper with relative degree d_r. We note from Eqs. (11) and (20) that $H_1(s)$ has the same denominator as $H(s)$, and it therefore follows from Eq. (9) that $H_1(s)$ is stable. Using similar arguments, it can be verified that $H_0(s)$ is proper and stable. Similarly, $H_2(s)$ is strictly proper and stable. Let

$$\Delta = \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_0(s)\|_{\mathcal{L}_1}(L\rho + L_0) + \bar{\gamma}\left(\|H_1(s)/M(s)\|_{\mathcal{L}_1} + L\|H_0(s)\|_{\mathcal{L}_1}\frac{\|H_2(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}\right) \quad (21)$$

where $\bar{\gamma} > 0$ is an arbitrary constant. Because both $H_1(s)$ and $M(s)$ are stable and strictly proper with relative degree d_r and $M(s)$ is minimum-phase, $H_1(s)/M(s)$ is stable and proper. Hence, $\|H_1(s)/M(s)\|_{\mathcal{L}_1}$ exists. Therefore, Δ is bounded. Further, because A_m is Hurwitz, there exists $P = P^\top > 0$, which satisfies the algebraic Lyapunov equation:

$$A_m^\top P + PA_m = -Q, \qquad Q > 0$$

From the properties of P, it follows that there exits nonsingular $\sqrt{P}$ such that

$$P = (\sqrt{P})^\top \sqrt{P}$$

Given the vector $c_m^\top(\sqrt{P})^{-1}$, let D be a $(n-1)\times n$ matrix that contains the null space of $c_m^\top(\sqrt{P})^{-1}$:

$$D(c_m^\top(\sqrt{P})^{-1})^\top = 0 \quad (22)$$

and further let

$$\Lambda = \begin{bmatrix} c_m^\top \\ D\sqrt{P} \end{bmatrix}$$

Lemma 2. For any

$$\xi = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$$

where $y \in \mathbb{R}$ and $z \in \mathbb{R}^{n-1}$, there exist $p_1 > 0$ and positive definite $P_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$\xi^\top(\Lambda^{-1})^\top P\Lambda^{-1}\xi = p_1 y^2 + z^\top P_2 z$$

Proof. Using $P = (\sqrt{P})^\top\sqrt{P}$, one can write

$$\xi^\top(\Lambda^{-1})^\top P\Lambda^{-1}\xi = \xi^\top(\sqrt{P}\Lambda^{-1})^\top(\sqrt{P}\Lambda^{-1})\xi$$

We note that

$$\Lambda(\sqrt{P})^{-1} = \begin{bmatrix} c_m^\top(\sqrt{P})^{-1} \\ D \end{bmatrix}$$

Let

$$q_1 = (c_m^\top(\sqrt{P})^{-1})(c_m^\top(\sqrt{P})^{-1})^\top, \qquad Q_2 = DD^\top$$

From Eq. (22), we have

$$(\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top = \begin{bmatrix} q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

Nonsingularity of Λ and $\sqrt{P}$ implies that

$$(\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top$$

is nonsingular, and therefore Q_2 is also nonsingular. Hence,

$$\left(\sqrt{P}\Lambda^{-1}\right)^\top\left(\sqrt{P}\Lambda^{-1}\right) = \left(\Lambda(\sqrt{P})^{-1}\right)\left(\Lambda(\sqrt{P})^{-1})^\top\right)^{-1} = \left(\Lambda(\sqrt{P})^{-1}\right)^{-\top}\left(\sqrt{P}\Lambda^{-1}\right) = \begin{bmatrix} q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix}$$

Denoting $p_1 = q_1^{-1}$ and $P_2 = Q_2^{-1}$ completes the proof. □

Let T be any positive constant, which can be associated with the sampling rate of the available CPU, and let $\mathbf{1}_1 \in \mathbb{R}^n$ be the basis vector with the first element equal to 1 and all other elements equal to 0. Let $\phi(T) \in \mathbb{R}^{n-1}$ be a vector, which consists of 2 to n elements of $\mathbf{1}_1^\top \exp[\Lambda A_m\Lambda^{-1}T]$, and let

$$\kappa(T) = \int_0^T |\mathbf{1}_1^\top \exp[\Lambda A_m\Lambda^{-1}(T-\tau)]\Lambda b_m|\mathrm{d}\tau \quad (23)$$

Further, let

$$\varsigma(T) = \|\phi(T)\|\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \kappa(T)\Delta$$
$$\alpha = \lambda_{\max}(\Lambda^{-\top}P\Lambda^{-1})\left(\frac{2\Delta\|\Lambda^{-\top}Pb_m\|}{\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})}\right)^2 \quad (24)$$

Letting

$$\mathbf{1}_1^\top \exp[\Lambda A_m\Lambda^{-1}t] = [\eta_1(t)\eta_2^\top(t)]$$

where $\eta_1(t) \in \mathbb{R}$ and $\eta_2(t) \in \mathbb{R}^{n-1}$ contain the first and 2 to n elements of the row vector $\mathbf{1}_1^\top\exp[\Lambda A_m\Lambda^{-1}t]$, respectively, we introduce the following functions:

$$\beta_1(T) = \max_{t\in[0,T]}|\eta_1(t)|, \qquad \beta_2(T) = \max_{t\in[0,T]}\|\eta_2(t)\| \quad (25)$$

Further, let $\Phi(T)$ be the $n\times n$ matrix:

$$\Phi(T) = \int_0^T \exp[\Lambda A_m\Lambda^{-1}(T-\tau)]\Lambda\mathrm{d}\tau \quad (26)$$

$$\beta_3(T) = \max_{t\in[0,T]}\eta_3(t), \qquad \beta_4(T) = \max_{t\in[0,T]}\eta_4(t) \quad (27)$$

where

$$\eta_3(t) = \int_0^t |\mathbf{1}_1^\top\exp[\Lambda A_m\Lambda^{-1}(t - \tau)]\Lambda\Phi^{-1}(T)\exp[\Lambda A_m\Lambda^{-1}T]\mathbf{1}_1|\mathrm{d}\tau,$$
$$\eta_4(t) = \int_0^t |\mathbf{1}_1^\top\exp[\Lambda A_m\Lambda^{-1}(t-\tau)]\Lambda b_m|\mathrm{d}\tau$$

Finally, let

$$\gamma_0(T) = \beta_1(T)\varsigma(T) + \beta_2(T)\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T)\varsigma(T) + \beta_4(T)\Delta \quad (28)$$

Lemma 3. The following limiting relationship is true:

$$\lim_{T\to 0}\gamma_0(T) = 0$$

Proof. Note that because $\beta_1(T)$, $\beta_3(T)\alpha$, and Δ are bounded, it is sufficient to prove that

$$\lim_{T\to 0}\varsigma(T) = 0 \quad (29)$$

$$\lim_{T\to 0}\beta_2(T) = 0 \quad (30)$$

$$\lim_{T\to 0}\beta_4(T) = 0 \quad (31)$$

Because

$$\lim_{T\to 0}\mathbf{1}_1^\top \exp[\Lambda A_m \Lambda^{-1} T] = \mathbf{1}_1^\top$$

then

$$\lim_{T\to 0}\phi(T) = \mathbf{0}_{n-1}$$

which implies

$$\lim_{T\to 0}\|\phi(T)\| = 0$$

Further, it follows from the definition of $\kappa(T)$ in Eq. (23) that

$$\lim_{T\to 0}\kappa(T) = 0$$

Because α and Δ are bounded,

$$\lim_{T\to 0}\varsigma(T) = 0$$

which proves Eq. (29). Because $\eta_2(t)$ is continuous, it follows from Eq. (25) that

$$\lim_{T\to 0}\beta_2(T) = \lim_{t\to 0}\|\eta_2(t)\|$$

Because

$$\lim_{t\to 0}\mathbf{1}_1^\top \exp[\Lambda A_m \Lambda^{-1} t] = \mathbf{1}_1^\top$$

we have

$$\lim_{t\to 0}\|\eta_2(t)\| = 0$$

which proves Eq. (30). Similarly,

$$\lim_{T\to 0}\beta_4(T) = \lim_{t\to 0}\|\eta_4(t)\| = 0$$

which proves Eq. (31). The boundedness of α, $\beta_3(T)$, and Δ implies

$$\lim_{T\to 0}\left(\beta_1(T)\varsigma(T) + \beta_2(T)\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T)\varsigma(T) + \beta_4(T)\Delta\right) = 0$$

which completes the proof. □

V. $\mathcal{L}_1$ Adaptive Output-Feedback Controller

We consider the following state predictor (or passive identifier):

$$\dot{\hat{x}}(t) = A_m\hat{x}(t) + b_m u(t) + \hat{\sigma}(t)$$
$$\hat{y}(t) = c_m^\top \hat{x}(t), \qquad \hat{x}(0) = x_0 = 0 \tag{32}$$

where $\hat{\sigma}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters. Note that although $\sigma(t) \in \mathbb{R}$ in Eq. (4) (i.e., the unknown disturbance) is matched, the uncertainty estimation in Eq. (32), $\hat{\sigma}(t) \in \mathbb{R}^n$, is unmatched. This is the key step of the solution and the subsequent analysis.

Letting $\tilde{y}(t) = \hat{y}(t) - y(t)$, the update law for $\hat{\sigma}(t)$ is given by

$$\hat{\sigma}(t) = \hat{\sigma}(iT), \qquad t \in [iT, (i+1)T)$$
$$\hat{\sigma}(iT) = -\Phi^{-1}(T)\mu(iT), \qquad i = 0, 1, 2, \dots \tag{33}$$

where $\Phi(T)$ is defined in Eq. (26) and

$$\mu(iT) = \exp[\Lambda A_m \Lambda^{-1} T]\mathbf{1}_1 \tilde{y}(iT), \qquad i = 0, 1, 2, 3, \dots \tag{34}$$

The control signal is defined as follows:

$$u(s) = C(s)r(s) - \frac{C(s)}{c_m^\top(s\mathbb{I} - A_m)^{-1}b_m} c_m^\top(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s) \tag{35}$$

where $C(s)$ was first introduced in Eq. (7). The $\mathcal{L}_1$ adaptive controller consists of Eqs. (32), (33), and (35), subject to the condition in Eq. (9).

We will now proceed with the computation of error bounds. Let $\tilde{x}(t) = \hat{x}(t) - x(t)$. The error dynamics between Eqs. (4) and (32) are

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + \hat{\sigma}(t) - b_m\sigma(t)$$
$$\tilde{y}(t) = c_m^\top\tilde{x}(t), \qquad \tilde{x}(0) = 0 \tag{36}$$

Lemma 4. Let $e(t) = y(t) - y_{\text{ref}}(t)$. Then

$$\|e_t\|_{\mathcal{L}_\infty} \le \frac{\|H_2(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L}\|\tilde{y}_t\|_{\mathcal{L}_\infty} \tag{37}$$

Proof. Let

$$\tilde{\sigma}(s) = \frac{C(s)}{c_m^\top(s\mathbb{I} - A_m)^{-1}b_m} c_m^\top(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s) - C(s)\sigma(s) \tag{38}$$

It follows from Eq. (35) that

$$u(s) = C(s)r(s) - C(s)\sigma(s) - \tilde{\sigma}(s) \tag{39}$$

and the system in Eq. (2) consequently takes the form

$$y(s) = M(s)(C(s)r(s) + (1 - C(s))\sigma(s) - \tilde{\sigma}(s)) \tag{40}$$

Substituting $u(s)$ from Eq. (39) into Eq. (3) gives

$$\sigma(s) = ((A(s) - M(s))(C(s)r(s) - C(s)\sigma(s) - \tilde{\sigma}(s)) + A(s)d(s))/M(s)$$

and hence,

$$\sigma(s) = \frac{(A(s) - M(s))(C(s)r(s) - \tilde{\sigma}(s)) + A(s)d(s)}{M(s) + C(s)(A(s) - M(s))} \tag{41}$$

Using the definitions of $H_0(s)$ and $H_1(s)$ in Eqs. (16) and (17), we can write

$$\sigma(s) = H_1(s)r(s) - \frac{H_1(s)}{C(s)}\tilde{\sigma}(s) + H_0(s)d(s) \tag{42}$$

Substitution into Eq. (40) leads to

$$y(s) = M(s)(C(s) + H_1(s)(1 - C(s)))\left(r(s) - \frac{\tilde{\sigma}(s)}{C(s)}\right) + H_0(s)M(s)(1 - C(s))d(s)$$

Recalling the definition of $H(s)$ from Eq. (8), one can verify that

$$M(s)(C(s) + H_1(s)(1 - C(s))) = H(s)C(s)$$

$$H(s) = H_0(s)M(s)$$

which implies

$$y(s) = H(s)(C(s)r(s) - \tilde{\sigma}(s)) + H(s)(1 - C(s))d(s)$$

Using the expression for $y_{\text{ref}}(s)$ from Eq. (14) and letting $d_e(s)$ be the Laplace transform of

$$d_e(t) = f(t, y(t)) - f(t, y_{\text{ref}}(t))$$

one can derive

$$e(s) = H(s)((1 - C(s))d_e(s) - \tilde{\sigma}(s))$$

Lemma 5 in the Appendix and Assumption 1 give the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \le L\|H(s)(1 - C(s))\|_{\mathcal{L}_1}\|e_t\|_{\mathcal{L}_\infty} + \|r_{1_t}\|_{\mathcal{L}_\infty} \tag{43}$$

where $r_1(t)$ is the signal, with its Laplace transform being

$$r_1(s) = H(s)\tilde{\sigma}(s)$$

Using the expression for $\tilde{\sigma}(s)$ from Eq. (38), along with the expression for $y(s)$ from Eq. (2), and taking into consideration that

$$\hat{y}(s) = M(s)u(s) + c_m^\top(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s)$$

we have

$$\begin{aligned}\tilde{y}(s) &= c_m^\top(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s) - M(s)\sigma(s)\\ &= \frac{M(s)}{C(s)}\frac{C(s)}{M(s)}c_m^\top(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s) - \frac{M(s)}{C(s)}C(s)\sigma(s)\\ &= \frac{M(s)}{C(s)}\tilde{\sigma}(s)\end{aligned} \tag{44}$$

This implies that $r_1(s)$ can be rewritten as

$$r_1(s) = \frac{C(s)H(s)}{M(s)}\frac{M(s)}{C(s)}\tilde{\sigma}(s) = H_2(s)\tilde{y}(s)$$

and hence

$$\|r_{1_t}\|_{\mathcal{L}_\infty} \leq \|H_2(s)\|_{\mathcal{L}_1}\|\tilde{y}_t\|_{\mathcal{L}_\infty}$$

Substituting this back into Eq. (43) completes the proof. □

VI. Analysis of $\mathcal{L}_1$ Adaptive Controller

In this section, we analyze the stability and the performance of the $\mathcal{L}_1$ adaptive controller. Using the definitions from Eq. (10), $H_3(s)$ in Eq. (19) can be rewritten as

$$H_3(s) = \frac{-C_n(s)A_d(s)M_n(s)}{H_d(s)} \tag{45}$$

where $H_d(s)$ is defined in Eq. (12). Because

$$\deg(C_d(s) - C_n(s)) - \deg C_n(s) = d_r$$

it can be checked straightforwardly that $H_3(s)$ is strictly proper. We note from Eqs. (11) and (45) that $H_3(s)$ has the same denominator as $H(s)$, and therefore it follows from Eq. (9) that $H_3(s)$ is stable. Because $H_3(s)$ is strictly proper and stable with relative degree d_r, $H_3(s)/M(s)$ is stable and proper and, therefore its $\mathcal{L}_1$ norm is finite. Consider the state transformation

$$\tilde{\xi} = \Lambda\tilde{x}$$

It follows from Eq. (36) that

$$\dot{\tilde{\xi}}(t) = \Lambda A_m\Lambda^{-1}\tilde{\xi}(t) + \Lambda\hat{\sigma}(t) - \Lambda b_m\sigma(t), \qquad \tilde{\xi}(0) = 0 \tag{46}$$

$$\tilde{y}(t) = \tilde{\xi}_1(t) \tag{47}$$

where $\tilde{\xi}_1(t)$ is the first element of $\tilde{\xi}(t)$.

Theorem 1. Given the system in Eq. (1) and the $\mathcal{L}_1$ adaptive controller in Eqs. (32), (33), and (35), subject to Eq. (9), if we choose T to ensure

$$\gamma_0(T) < \bar{\gamma} \tag{48}$$

where $\bar{\gamma}$ is an arbitrary positive constant introduced in Eq. (21), then

$$\|\tilde{y}\|_{\mathcal{L}_\infty} < \bar{\gamma} \tag{49}$$

$$\|y - y_{\rm ref}\|_{\mathcal{L}_\infty} \leq \gamma_1, \qquad \|u - u_{\rm ref}\|_{\mathcal{L}_\infty} \leq \gamma_2 \tag{50}$$

with

$$\gamma_1 = \|H_2(s)\|_{\mathcal{L}_1}\bar{\gamma}/(1 - \|G(s)\|_{\mathcal{L}_1}L)$$
$$\gamma_2 = L\|H_2(s)\|_{\mathcal{L}_1}\gamma_1 + \|H_3(s)/M(s)\|_{\mathcal{L}_1}\bar{\gamma}$$

Proof. First, we prove the bound in Eq. (49) by a contradiction argument. Because $\tilde{y}(0) = 0$ and $\tilde{y}(t)$ is continuous, then assuming the opposite implies that there exists t' such that

$$\|\tilde{y}(t)\| < \bar{\gamma}, \quad \forall\ 0 \leq t < t' \tag{51}$$

$$\|\tilde{y}(t')\| = \bar{\gamma} \tag{52}$$

which leads to

$$\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} = \bar{\gamma} \tag{53}$$

Because $y(t) = y_{\rm ref}(t) + e(t)$, the upper bound in Eq. (15) can be used to arrive at

$$\begin{aligned}\|y_{t'}\|_{\mathcal{L}_\infty} &\leq \|y_{{\rm ref}_{t'}}\|_{\mathcal{L}_\infty} + \|e_{t'}\|_{\mathcal{L}_\infty} \leq \rho\\ &+ \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\bar{\gamma}/(1 - \|G(s)\|_{\mathcal{L}_1}L)\end{aligned} \tag{54}$$

It follows from Eqs. (42) and (44) that

$$\sigma(s) = H_1(s)r(s) - H_1(s)\tilde{y}(s)/M(s) + H_0(s)d(s)$$

and hence Eq. (53) implies that

$$\begin{aligned}\|\sigma_{t'}\|_{\mathcal{L}_\infty} &\leq \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_1(s)/M(s)\|_{\mathcal{L}_1}\bar{\gamma}\\ &+ \|H_0(s)\|_{\mathcal{L}_1}(L\|y_{t'}\|_{\mathcal{L}_\infty} + L_0)\end{aligned}$$

which, along with Eq. (54), leads to

$$\|\sigma_{t'}\|_{\mathcal{L}_\infty} \leq \Delta \tag{55}$$

It follows from Eq. (46) that

$$\begin{aligned}\tilde{\xi}(iT + t) &= \exp[\Lambda A_m\Lambda^{-1}t]\tilde{\xi}(iT)\\ &+ \int_{iT}^{iT+t}\exp[\Lambda A_m\Lambda^{-1}(iT + t - \tau)]\Lambda\hat{\sigma}(iT){\rm d}\tau\\ &- \int_{iT}^{iT+t}\exp[\Lambda A_m\Lambda^{-1}(iT + t - \tau)]\Lambda b_m\sigma(\tau){\rm d}\tau\\ &= \exp[\Lambda A_m\Lambda^{-1}t]\tilde{\xi}(iT) + \int_0^t\exp[\Lambda A_m\Lambda^{-1}(t - \tau)]\Lambda\hat{\sigma}(iT){\rm d}\tau\\ &- \int_0^t\exp[\Lambda A_m\Lambda^{-1}(t - \tau)]\Lambda b_m\sigma(iT + \tau){\rm d}\tau\end{aligned} \tag{56}$$

Because

$$\tilde{\xi}(iT) = \begin{bmatrix}\tilde{y}(iT)\\ 0\end{bmatrix} + \begin{bmatrix}0\\ \tilde{z}(iT)\end{bmatrix}$$

it follows from Eq. (56) that

$$\tilde{\xi}(iT + t) = \chi(iT + t) + \zeta(iT + t) \tag{57}$$

where

$$\begin{aligned}\chi(iT + t) = \exp[\Lambda A_m\Lambda^{-1}t]\begin{bmatrix}\tilde{y}(iT)\\ 0\end{bmatrix} +\\ \int_0^t\exp[\Lambda A_m\Lambda^{-1}(t - \tau)]\Lambda\hat{\sigma}(iT){\rm d}\tau\end{aligned} \tag{58}$$

$$\begin{aligned}\zeta(iT + t) &= \exp[\Lambda A_m\Lambda^{-1}t]\begin{bmatrix}0\\ \tilde{z}(iT)\end{bmatrix}\\ &- \int_0^t\exp[\Lambda A_m\Lambda^{-1}(t - \tau)]\Lambda b_m\sigma(iT + \tau){\rm d}\tau\end{aligned} \tag{59}$$

In what follows, we prove that for all $iT \leq t'$, one has

$$|\tilde{y}(iT)| \leq \varsigma(T) \tag{60}$$

$$\tilde{z}^\top(iT)P_2\tilde{z}(iT) \leq \alpha \tag{61}$$

where $\varsigma(T)$ and α are defined in Eq. (24). Because $\tilde{\xi}(0) = 0$, it is straightforward that $|\tilde{y}(0)| \leq \varsigma(T)$ and $\tilde{z}^\top(0)P_2\tilde{z}(0) \leq \alpha$. For any $(j+1)T \leq t'$, we will prove that if

$$|\tilde{y}(jT)| \leq \varsigma(T) \tag{62}$$

$$\tilde{z}^\top(jT)P_2\tilde{z}(jT) \leq \alpha \tag{63}$$

then Eqs. (62) and (63) hold for $j+1$ too. Hence, Eqs. (60) and (61) hold for all $iT \leq t'$.

Assume that Eqs. (62) and (63) hold for j and, in addition,

$$(j+1)T \leq t' \tag{64}$$

It follows from Eq. (57) that

$$\tilde{\xi}((j+1)T) = \chi((j+1)T) + \zeta((j+1)T) \tag{65}$$

where

$$\chi((j+1)T) = \exp[\Lambda A_m \Lambda^{-1} T]\begin{bmatrix} \tilde{y}(jT) \\ 0 \end{bmatrix} + \int_0^T \exp[\Lambda A_m \Lambda^{-1}(T-\tau)]\Lambda\hat{\sigma}(jT)\mathrm{d}\tau \tag{66}$$

$$\zeta((j+1)T) = \exp[\Lambda A_m \Lambda^{-1} T]\begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} - \int_0^T \exp[\Lambda A_m \Lambda^{-1}(T-\tau)]\Lambda b_m \sigma(jT+\tau)\mathrm{d}\tau \tag{67}$$

Substituting the adaptive law from Eq. (33) in Eq. (66), we have

$$\chi((j+1)T) = 0 \tag{68}$$

It follows from Eq. (59) that $\zeta(t)$ is the solution of the following dynamics:

$$\dot{\zeta}(t) = \Lambda A_m \Lambda^{-1}\zeta(t) - \Lambda b_m \sigma(t) \tag{69}$$

$$\zeta(jT) = \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix}, \qquad t \in [jT, (j+1)T] \tag{70}$$

Consider the following function

$$V(t) = \zeta^\top(t)\Lambda^{-\top}P\Lambda^{-1}\zeta(t)$$

over $t \in [jT, (j+1)T]$. Because Λ is nonsingular and P is positive definite, $\Lambda^{-\top}P\Lambda^{-1}$ is positive definite, and hence $V(t)$ is a positive definite function. It follows from Lemma 2 and the relationship in Eq. (70) that

$$V(\zeta(jT)) = \tilde{z}^\top(jT)P_2\tilde{z}(jT)$$

which further, along with the upper bound in Eq. (63), leads to the following:

$$V(\zeta(jT)) \leq \alpha \tag{71}$$

It follows from Eq. (69) that over $t \in [jT, (j+1)T]$,

$$\begin{aligned}\dot{V}(t) &= \zeta^\top(t)\Lambda^{-\top}P\Lambda^{-1}\Lambda A_m\Lambda^{-1}\zeta(t) \\ &\quad + \zeta^\top(t)\Lambda^{-\top}A_m^\top\Lambda^\top\Lambda^{-\top}P^\top\Lambda^{-1}\zeta(t) - 2\zeta^\top(t)\Lambda^{-\top}P\Lambda^{-1}\Lambda b_m\sigma(t) \\ &= -\zeta^\top(t)\Lambda^{-\top}Q\Lambda^{-1}\zeta(t) - 2\zeta^\top(t)\Lambda^{-\top}Pb_m\sigma(t)\end{aligned}$$

Using the upper bound from Eq. (55), one can derive over $t \in [jT, (j+1)T]$

$$\dot{V}(t) \leq -\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})\|\zeta(t)\|^2 + 2\|\zeta(t)\|\|\Lambda^{-\top}Pb_m\|\Delta \tag{72}$$

Note that for all $t \in [jT, (j+1)T]$, if

$$V(t) \geq \alpha \tag{73}$$

we have

$$\|\zeta(t)\| \geq \sqrt{\frac{\alpha}{\lambda_{\max}(\Lambda^{-\top}P\Lambda^{-1})}} = \frac{2\Delta\|\Lambda^{-\top}Pb_m\|}{\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})}$$

and the upper bound in Eq. (72) yields

$$\dot{V}(t) \leq 0 \tag{74}$$

It follows from Eqs. (71), (73), and (74) that

$$V(t) \leq \alpha, \quad \forall\ t \in [jT, (j+1)T]$$

and therefore

$$V((j+1)T) = \zeta^\top((j+1)T)(\Lambda^{-\top}P\Lambda^{-1})\zeta((j+1)T) \leq \alpha \tag{75}$$

Because

$$\tilde{\xi}((j+1)T) = \chi((j+1)T) + \zeta((j+1)T) \tag{76}$$

the equality in Eq. (68) and the upper bound in Eq. (75) lead to the following inequality:

$$\tilde{\xi}^\top((j+1)T)(\Lambda^{-\top}P\Lambda^{-1})\tilde{\xi}((j+1)T) \leq \alpha$$

Using the result of Lemma 2, one can derive that

$$\begin{aligned}&\tilde{z}^\top((j+1)T)P_2\tilde{z}((j+1)T) \\ &\quad \leq \tilde{\xi}^\top((j+1)T)(\Lambda^{-\top}P\Lambda^{-1})\tilde{\xi}((j+1)T) \leq \alpha\end{aligned}$$

which implies that the upper bound in Eq. (63) holds for $j+1$.

It follows from Eqs. (47), (68), and (76), that

$$\tilde{y}((j+1)T) = \mathbf{1}_1^\top\zeta((j+1)T)$$

and the definition of $\zeta((j+1)T)$ in Eq. (67) leads to the following expression:

$$\tilde{y}((j+1)T) = \mathbf{1}_1^\top \exp[\Lambda A_m\Lambda^{-1}T]\begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} - \mathbf{1}_1^\top\int_0^T \exp[\Lambda A_m\Lambda^{-1}(T-\tau)]\Lambda b_m\sigma(jT+\tau)\mathrm{d}\tau \tag{77}$$

The upper bounds in Eqs. (55) and (63) allow for the following upper bound:

$$\begin{aligned}|\tilde{y}((j+1)T)| &\leq \|\phi(T)\|\|\tilde{z}(jT)\| \\ &\quad + \int_0^T |\mathbf{1}_1^\top \exp[\Lambda A_m\Lambda^{-1}(T-\tau)]\Lambda b_m||\sigma(jT+\tau)|\mathrm{d}\tau \\ &\leq \|\phi(T)\|\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \kappa(T)\Delta = \varsigma(T)\end{aligned} \tag{78}$$

where $\phi(T)$ and $\kappa(T)$ are defined in Eq. (23), and $\varsigma(T)$ is defined in Eq. (24). This confirms the upper bound in Eq. (62) for $j+1$. Hence, Eqs. (60) and (61) hold for all $iT \leq t'$.

For all $iT + t \leq t'$, where $0 \leq t \leq T$, using the expression from Eq. (56), we can write that

$$\tilde{y}(iT+t) = \mathbf{1}_1^\top \exp[\Lambda A_m \Lambda^{-1} t]\tilde{\xi}(iT) + \mathbf{1}_1^\top \int_0^t \exp[\Lambda A_m \Lambda^{-1}(t-\tau)]\Lambda\hat{\sigma}(iT)\mathrm{d}\tau - \mathbf{1}_1^\top \int_0^t \exp[\Lambda A_m \Lambda^{-1}(t-\tau)]\Lambda b_m \sigma(iT+\tau)\mathrm{d}\tau$$

The upper bound in Eq. (55) and definitions of $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$, and $\eta_4(t)$ allow for the following upper bound:

$$|\tilde{y}(iT+t)| \le |\eta_1(t)||\tilde{y}(iT)| + \|\eta_2(t)\|\|\tilde{z}(iT)\| + \eta_3(t)|\tilde{y}(iT)| + \eta_4(t)\Delta$$

Taking into consideration Eqs. (62) and (63) and recalling the definitions of $\beta_1(T)$, $\beta_2(T)$, $\beta_3(T)$, and $\beta_4(T)$ in Eqs. (25–27) for all $0 \le t \le T$ and for any nonnegative integer i subject to $iT + t \le t'$, we have

$$|\tilde{y}(iT+t)| \le \beta_1(T)\varsigma(T) + \beta_2(T)\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T)\varsigma(T) + \beta_4(T)\Delta$$

Because the right-hand side coincides with the definition of $\gamma_0(T)$ in Eq. (28), then for all $t \in [0, t']$, we have

$$|\tilde{y}(t)| \le \gamma_0(T)$$

which, along with the assumption on T introduced in Eq. (48), yields

$$\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} < \bar{\gamma}$$

This clearly contradicts the statement in Eq. (53). Therefore,

$$\|\tilde{y}\|_{\mathcal{L}_\infty} < \bar{\gamma}$$

which proves Eq. (49). Further, it follows from Lemma 4 that

$$\|e_t\|_{\mathcal{L}_\infty} \le \frac{\|H_2(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L}\bar{\gamma}$$

which holds uniformly for all $t \ge 0$ and therefore leads to the first upper bound in Eq. (50).

It follows from Eqs. (39) and (41) that

$$u(s) = \frac{M(s)C(s)r(s) - M(s)\tilde{\sigma}(s) - C(s)A(s)d(s)}{C(s)A(s) + (1 - C(s))M(s)}$$

To prove the second bound in Eq. (50), we use the expression of $u_{\mathrm{ref}}(s)$ from Eq. (13) to derive

$$u(s) - u_{\mathrm{ref}}(s) = -H_2(s)d_e(s) + \frac{H_3(s)}{C(s)}\tilde{\sigma}(s) = -H_2(s)d_e(s) + \frac{H_3(s)}{M(s)}\frac{M(s)}{C(s)}\tilde{\sigma}(s) \tag{79}$$

It follows from Eqs. (44) and (79) that

$$\|u - u_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le L\|H_2(s)\|_{\mathcal{L}_1}\|y - y_{\mathrm{ref}}\|_{\mathcal{L}_\infty} + \|H_3(s)/M(s)\|_{\mathcal{L}_1}\|\tilde{y}\|_{\mathcal{L}_\infty}$$

which, along with Eqs. (49) and (50), leads to the second bound in Eq. (50). The proof is complete. □

Thus, the tracking error between $y(t)$ and $y_{\mathrm{ref}}(t)$, as well between $u(t)$ and $u_{\mathrm{ref}}(t)$, is uniformly bounded by a constant proportional to T. This implies that during the transient one can achieve arbitrary improvement of tracking performance by uniformly reducing T.

Remark 1. Note that the parameter T is the fixed time step in the definition of the adaptive law. The adaptive parameters in $\hat{\sigma}(t) \in \mathbb{R}^n$ take constant values during $[iT, (i+1)T)$ for every $i = 0, 1, \ldots$. Reducing T imposes hardware (CPU) requirements, and Theorem 1 further implies that the performance limitations are consistent with the hardware limitations. This in turn is consistent with the earlier results in [5,6], in which improvement of the transient performance was achieved by increasing the adaptation rate in the continuous-time adaptive laws.

Remark 2. We note that the following ideal control signal

$$u_{\mathrm{ideal}}(t) = r(t) - \sigma_{\mathrm{ref}}(t)$$

is the one that leads to desired system response

$$y_{\mathrm{ideal}}(s) = M(s)r(s)$$

by canceling the uncertainties exactly. Thus, the reference system in Eqs. (5–7) has a different response from the ideal one. It only cancels the uncertainties within the bandwidth of $C(s)$, which can be selected to be compatible with the control channel specifications. This is exactly what one can hope to achieve with any feedback in the presence of uncertainties.

Remark 3. We note that stability of $H(s)$ is equivalent to stabilization of $A(s)$ by

$$\frac{C(s)}{M(s)(1 - C(s))} \tag{80}$$

Indeed, consider the closed-loop system, composed of the system $A(s)$ and negative feedback of Eq. (80). The closed-loop transfer function is

$$A(s)\Big/\left[1 + A(s)\frac{C(s)}{M(s)(1 - C(s))}\right] \tag{81}$$

Incorporating Eq. (10), one can verify that the denominator of the system in Eq. (81) is exactly $H_d(s)$. Hence, stability of $H(s)$ is equivalent to the stability of the closed-loop system in Eq. (81). This implies that the class of systems $A(s)$ that can be stabilized by the $\mathcal{L}_1$ adaptive output-feedback controller Eqs. (32), (33), and (35) is not empty.

Remark 4. Finally, we note that although the feedback in Eq. (80) may stabilize the system in Eq. (1) for some classes of unknown nonlinearities, it will not ensure uniform transient performance in the presence of unknown $A(s)$. On the contrary, the $\mathcal{L}_1$ adaptive controller ensures uniform transient performance for both of the system's signals, independent of the unknown nonlinearity and independent of $A(s)$.

VII. Simulations

A. Numerical Example

Consider the system in Eq. (1) with

$$A(s) = (s+1)/(s^3 - s^2 - 2s + 8)$$

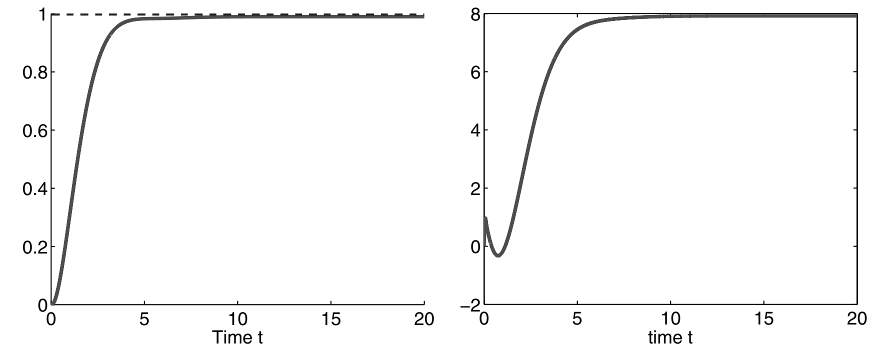
We note that $A(s)$ has poles in the right-half complex plane. We consider the $\mathcal{L}_1$ adaptive output-feedback controller defined via Eqs. (32), (33), and (35), where

$$M(s) = \frac{1}{s^2 + 1.4s + 1}$$
$$C(s) = \frac{100}{s^2 + 14s + 100}, \qquad T = 10^{-4}$$

First, we consider the step response for $d(t) = 0$. The simulation results of $\mathcal{L}_1$ adaptive controller are shown in Figs. 1a and 1b. Next, we consider

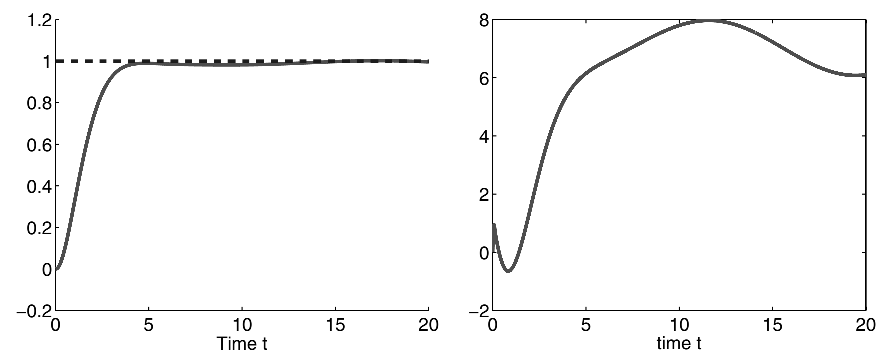
$$d(t) = f(t, y(t)) = \sin(0.1t)y^2(t) + \sin(0.4t)$$

and apply the same controller without retuning. The control signal and the system response are plotted in Figs. 2a and 2b. Further, we consider a time-varying reference input $r(t) = 0.5\sin(0.3t)$ and note that without any retuning of the controller, the system response and the control signal behave as expected (Figs. 3a and 3b).



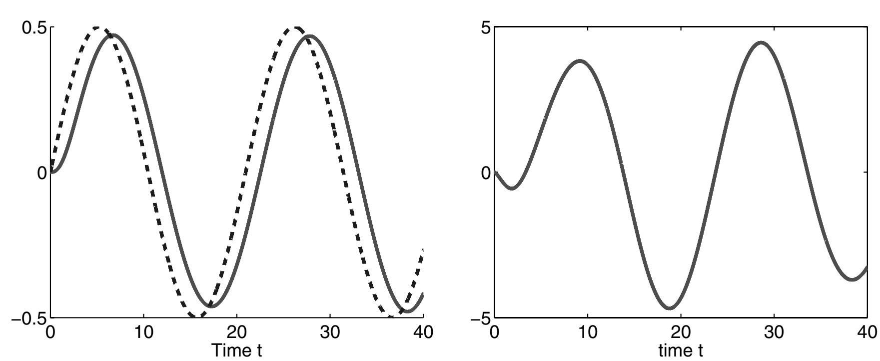
a) $y(t)$ **(solid) and** $r(t)$ **(dashed)** **b) Time history of** $u(t)$

Fig. 1 Performance for $r(t) = 1$ **and** $d(t) = 0$**.**



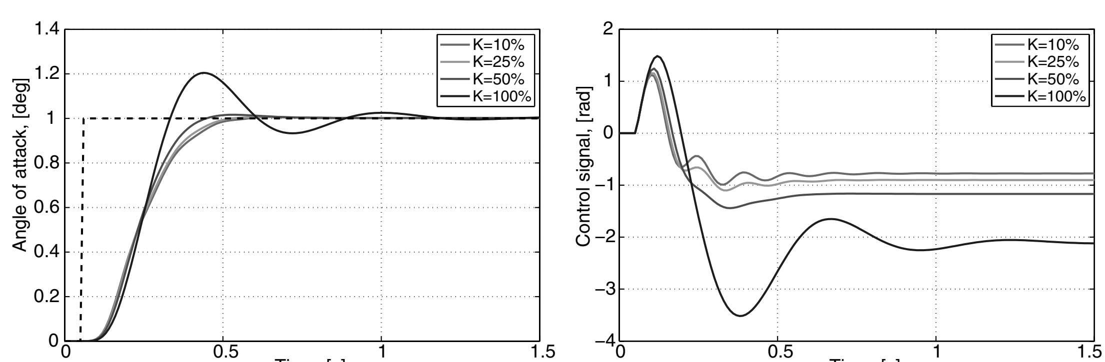
a) $y(t)$ **(solid) and** $r(t)$ **(dashed)** **b) Time history of** $u(t)$

Fig. 2 Performance for $r(t) = 1$ **and** $d(t) = \sin(0.1t)y^2(t) + \sin(0.4t)$**.**



a) $y(t)$ **(solid) and** $r(t)$ **(dashed)** **b) Time history of** $u(t)$

Fig. 3 Performance for $r(t) = 0.5\sin(0.3t)$ **and** $d(t) = \sin(0.1t)y^2(t) + \sin(0.4t)$**.**



a) System response **b) Control history**

Fig. 4 Simulation results for missile longitudinal autopilot design: Scaled response for scaled (unmatched) uncertainties.

B. Missile Longitudinal Autopilot Design

We consider the missile dynamics from [38], which are given by the following state-space representation:

$$\dot{x}(t) = \begin{bmatrix} \frac{Z_a}{V} & 1 & \frac{Z_d}{V} & 0 \\ M_a & 0 & M_d & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_a^2 & -2\zeta_a\omega_a \end{bmatrix} x(t) + \Delta A x(t)$$
$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega_a^2 \end{bmatrix} (u(t) + k_m^\top x(t))$$
$$y(t) = [1 \quad 0 \quad 0 \quad 0] x(t)$$

where $x(t) = [A_z \quad q \quad \delta_a \quad \dot{\delta}_a]^\top$ is the system state, in which A_z (ft/s^2) is the vertical acceleration, q (rad/s) is the pitch rate, δ_a (rad) is the fin deflection, $\dot{\delta}_a$ (rad/s) is the fin deflection rate, k_m is the vector of matched (in the sense of state-feedback) parametric uncertainties, and ΔA contains unmatched uncertainties. For detailed analysis of the missile dynamics, the reader is referred to [38].

In simulations, the following numerical values have been used for the missile dynamics [38]:

$$\frac{Z_a}{V} = -1.3046\left[\frac{1}{s}\right], \qquad \frac{Z_d}{V} = -0.2142\left[\frac{1}{s}\right]$$
$$M_a = 47.7109\left[\frac{1}{s^2}\right], \qquad M_d = -104.8346\left[\frac{1}{s^2}\right]$$
$$\Delta A = \begin{bmatrix} -0.4996K & 0 & -0.4996K & 0 \\ 0.4996K & 0 & -0.4996K & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$k_m = 0.3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ -2\frac{\zeta_a}{\omega_a} \end{bmatrix}$$

Here, the matrix ΔA is selected to enforce the worst direction for the change of aerodynamic parameters, with K being the uncertainty scaling factor for unmatched uncertainties. In this example, $K = 0$ corresponds to nonperturbed aerodynamics, and the value $K = 1$ corresponds to uncertainties that would turn the system into an oscillator in the presence of an linear quadratic regulator state-feedback controller only.

For simulation of the $\mathcal{L}_1$ adaptive output-feedback controller, the following $M(s)$ and $C(s)$ have been selected, along with a sampling rate $T = 0.0001$ s:

$$M(s) = \frac{1}{1/\omega s^2 + 2\zeta\omega s + 1}, \qquad C(s) = \frac{1}{1/\omega_c s^2 + 2\zeta_c\omega_c s + 1}$$

where $\omega = 8$ rad/s, $\zeta = 0.9$, $\omega_c = 120$ rad/s, and $\zeta_c = 1.85$. Figures 4a and 4b show the scaled response for scaled uncertainties, dependent upon the changes in K, achieved via scaled control efforts.

VIII. Conclusions

We presented the $\mathcal{L}_1$ adaptive output-feedback controller for reference systems that do not verify the SPR condition for their input–output transfer function. The new piecewise constant adaptive law, along with the low-pass-filtered control signal, ensures uniform performance bounds for the system's input/output signals simultaneously. The performance bounds can be systematically improved by reducing the integration time step.

Appendix: Facts from Linear Systems Theory

Definition 1 [41]. For a signal $\xi(t)$, where $t \geq 0$ and $\xi \in \mathbb{R}^n$, its $\mathcal{L}_\infty$ and truncated $\mathcal{L}_\infty$ norms are

$$\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,\ldots,n}\left(\sup_{\tau\geq 0}|\xi_i(\tau)|\right)$$

$$\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,\ldots,n}\left(\sup_{0\leq\tau\leq t}|\xi_i(\tau)|\right)$$

where ξ_i is the ith component of ξ.

Definition 2 [41]. The $\mathcal{L}_1$ gain of a stable proper SISO system is defined:

$$||H(s)||_{\mathcal{L}_1} = \int_0^\infty |h(t)|\mathrm{d}t$$

where $h(t)$ is the impulse response of $H(s)$.

Lemma 5 [41]. For a stable proper multi-input/multi-output system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have

$$\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} \quad \forall\ t \geq 0$$

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